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Proper Use of the Modified Sharpe Ratios in Performance Measurement: Rearranging the Cornish Fisher Expansion

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Abstract Performance analysis is a key process in finance to evaluate or compare investment opportunities, allocations, or management. The classical method is to compute the market or sub-market returns and volatilities, and then calculate the standard performance measure, namely, the Sharpe ratio. This measure is based on the first two moments of a return distribution. Therefore, a significant weakness of this method is that it implicitly assumes that the distribution is Gaussian (if it is not Gaussian, the approach may lead to a bad fit for the distribution). In fact, risk comes from not only volatility, but also higher moments of distribution such as skewness and kurtosis. The standard method to resolve this issue is to use the modified Sharpe ratio; this method replaces the classical Sharpe ratio volatility with the value at risk. The latter is computed using the Cornish Fisher expansion, a tool based on the first four moments of return distribution. This methodology, however, may present a major pitfall: in some cases, quantile functions do not stay monotone. In this paper, we show how this tool can be used effectively through a specific procedure, rearrangement. We compare various metrics using rank correlation, and demonstrate how and in which cases the proposed procedure delivers ranking different from the standard Sharpe ratio ranking. Furthermore, we show how our technique offers better distribution approximations and is therefore a more useful performance metric. Institutional investors may find the technique proposed here useful in that it allows for considering non-normality in performance analysis.

Keywords: Performance measures; Cornish Fisher expansion; modified Sharpe ratio, rearrangement

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1 Introduction

Most financial performance analyses focus mostly on total return figures. The concern with using absolute returns comes from the structure of the investment industry. Indeed, most investment managers market their funds through brokers and investment advisors, whose funding is mostly based on the latest absolute returns performance. Thus, funds with the highest returns will raise more capital than those lagging behind. Fund managers are therefore highly incented to exhibit high absolute returns. Brokers and investment advisors for the most part do not act differently. They mostly do not discuss risk because of the nature of their commission-based compensation structure and most of them are paid only when the sale of a financial product is completed. This approach does not consider the link between return and risk, in contrast to Markowitz (1952). Nevertheless, only absolute returns are still widely compared in the field. Thus, in terms of management and allocations, absolute returns are superior to competition.

Investments can be compared in a straightforward manner only in very special cases; for example, when risks are equal, a higher expected return is always preferable, but when the expected returns are equal, a lower risk is always preferable. These matters could become complicated if we consider two or more markets with different expected returns and risks. Undoubtedly, such naïve comparisons would become extremely misleading if the funds or indices exhibit different risk characteristics. Investors often evaluate the performance of different portfolio strategies in order to compare or rank them. Thus, investment by nature is a two-dimensional process based on returns as well as the risk taken to achieve those returns. Specifically, given that a higher return (risk) is always (never) desirable, the next question is what additional return would be a sufficient compensation for an additional risk. This is precisely where risk-adjusted performance measures become helpful. Intuitively, a performance measure should consider the "reward," or upside potential, as well as the strategic risk, which has to be evaluated.

Combining the return and risk into a single useful risk-adjusted number is one of the strategic performance measurement tasks. Performance measures give the ratios of reward over risk. Basically, a good risk measurement must be able to compare the performance of markets with similar risk characteristics, as well as the performance of other funds with different risk characteristics. Even for a large number of performance measures, the Sharpe ratio would practically be the most commonly used risk-adjusted performance metric measure. Defined by the Nobel laureate Sharpe (1966), the Sharpe ratio measures the "excess return per unit of volatility." It is calculated by dividing the excess return of a market or fund by its volatility. Algebraically, it can be given by

$$SR = \frac{r_P - r_f}{\sigma_P}, \tag{1}$$

where r_P is the average return of portfolio P, r_f is the risk-free rate, [1] and σ_P is the standard deviation of the portfolio return P.[2] By analyzing this risk-adjusted performance ratio, we can identify the markets or categories that outperformed others. However, this method has an important weakness: it presupposes the normal return distribution since it considers only the first two moments of distribution.

When asset returns can reasonably be considered normally distributed, the Sharpe ratio would be useful because the distribution of returns will then be completely described by its mean and volatility (inputs to equation 1).[3] In case the return distribution cannot be considered as normal (such as a hedge fund, real estate, or option), the performance measures based on non-normality will have to be considered. In an attempt to solve the non-normality issue, Favre & Galeano (2002) and Gregoriou & Gueyie (2003) modified the traditional Sharpe ratio, to obtain the modified Sharpe ratio (mSR).[4] This is defined as the ratio of the excess return of a market, an asset, or a fund to its Value at -Risk, or VaR, where VaR is computed using the Cornish Fisher (CF) expansion and denoted as $mVaR$ (the CF expansion is defined in section).

Informally, VaR is the largest percentage loss with a given probability (confidence level) likely to be suffered on a portfolio position over a given holding period. In other words, for a given portfolio and time horizon, with the selected confidence level $\alpha \in (0, 1)$, VaR is defined as the threshold value, assuming no further trade, at which the probability of the mark-to-market loss in the portfolio exceeding this VaR level is exactly the preset probability of loss α.[5] Thus, VaR is the quantile of the projected distribution of losses over the target horizon, in that if α is taken as the confidence level, VaR will correspond to the α quantile. By convention, this worst loss is always expressed as a positive percentage in the manner indicated. Thus, in formal terms, if we take L to be the loss, measured as a positive number, and α to be the confidence level, then VaR can be defined as the smallest loss—in absolute terms—such that

$$P(L > VaR) \leq \alpha. \tag{2}$$

[1] The choice of risk-free rate has recently been debated. We do not discuss this point here because it is another topic.

[2] As the numbers are typically expressed on an annual basis, the Sharpe ratio is also expressed on an annual basis (particularly because the increases in standard deviation are not linear).

[3] From recent evidence, the Sharpe ratio can give fund rankings almost identical to alternative performance measures (see Eling & Schuhmacher, 2007).

[4] This performance metric is part of the $RhoVaR$ performance metric class

$$\rho VaR_\alpha = \frac{r_P - r_f}{VaR_\alpha},$$

,defined as the ratio of the excess return of a market, an asset, or a fund to its Value at Risk.

[5] Note that VaR provides no information on the likely severity of the loss by which its level will be exceeded.

A more detailed definition of VaR can be found in Jorion (2007) or Christoffersen (2012). [6]

Over the past few years, the popularity of downside risk measures (including VaR) has been growing. Today, these metrics are replacing standard deviation to evaluate the risk of investments. The reason behind the growing interest in downside risk measures is the decision of several regulators (Basel and Solvency) to rely almost solely on downside risk metrics such as VaR or its derivative the expected shortfall for calculation of the required capital.[7]

VaR is mainly estimated using one of the following three methods: historical, parametric (variance–covariance), and Monte Carlo simulations. The advantages and drawbacks of these methods are extensively discussed in (Christoffersen, 2012). The $mVaR$ in this work is computed using CF expansion.

The modified Sharpe ratio, denoted as mSR_α in the literature, where α is the probability level, is defined as

$$mSR_\alpha = \frac{r_P - r_f}{mVaR_\alpha}. \tag{3}$$

Here, r_P is the average portfolio return, r_f is the risk-free rate, and $mVaR$ is the way we compute VaR, which relies neither on strong assumptions (such as the need for normal distribution) nor on excessive data (as do historical methods). This is based on the CF expansion. This approach enables us to approximate the true (unknown) return distribution. It takes the form of a Gaussian quantile estimation plus some correction terms based on the return distribution's skewness and kurtosis. $mVaR$ is a convenient and useful tool for practitioners and academics owing to its precision and explicit form. Moreover, it is straightforward to compute and interpret. Thus, CF expansion is a relatively easy and parsimonious approach to deal with non-normality in asset prices or returns. The use of mSR clearly shows that risk comes from volatility as well as higher moments such as skewness and kurtosis (an overview of skewness and kurtosis parameters is provided in Appendix A). In particular, this study shows why this measure is preferable to the more traditional one

[6] In terms of gain rather than loss, the VaR at the confidence level α for the market rate of return X whose distribution function is denoted as $F_X(x) \equiv P[X \leq x]$, and whose quantile at level α is denoted as $q_\alpha(X)$, is

$$-VaR_\alpha(X) = \sup\{x : F_X(x) \leq \alpha\} \equiv q_\alpha(X).$$

[7] Note that all risk metrics are broadly criticized for their reliance on past returns and data. Dario Cintioli, who heads the risk analysis at StatPro, says that "the main problem with VaR, or any risk measure, is that you're using historical data in some way." Indeed, a conflict arises when choosing how much data to analyze. When excessive data are chosen, (from past experience) the results may be irrelevant. The opposite may be true when we consider recent history.

when properly used.

Even though $mVaR$ is a popular and useful technique, its definition domain (see section) restrains its use. Typically (and unfortunately), the literature does not specify the CF expansion's (see for instance Lee & Higgins, 2009; Christoffersen, 2012; Hull, 2012) definition domain, and so this could become a possible pitfall that must be addressed. Thus, CF expansion is generally restricted to cases where the distribution is close to normal. A solution to this issue has been proposed by Chernozhukov et al. (2010b). Their methodology enables the proper computation of $mVaRs$, irrespective of the distributions (see Amédée-Manesme et al., 2015).

This study builds on the literature as follows. Eling & Schuhmacher (2007) and Pedersen & Rudholm-Alfvin (2003) use hedge fund returns to discuss the suitability of classic and newer measures to compare hedge funds (see section 2), but we analyze simulated returns and show how to properly use the modified Sharpe ratio. Our findings can be summarized as follows. First, even when the standard Sharpe ratio gives a virtually identical rank ordering, with a light deviation in return distribution from the Gaussian, the modified Sharpe ratio becomes unavoidable as soon as the distribution deviates from the normal. Second, the rearrangement procedure should be adopted when the threshold is low (below 1%) and/or the distribution is asymmetric.

The remainder of this paper is organized as follows. Section 2 presents a literature review, while section 3 presents the CF expansion, its pitfall, and the solution, emphasizing on the proper use of this tool. Section 4 implements the proposed methodology and discusses the empirical results.

2 Literature review

The literature on performance measures has largely focused on examining whether or not managers can beat the market (Markowitz, 1952; Sharpe, 1966; Jensen, 1968; Malkiel & Fama, 1970). When maket distributions are symmetric and the classical mean-variance capital asset pricing model (CAPM) is valid, performance measures can be computed directly. The most commonly employed performing measure is the Sharpe ratio (Sharpe, 1966)[8] (and its derivation of the Treynor ratio developed by Treynor, 1965). Another common approach is Jensen's measure, or Jensen's alpha (developped by Jensen, 1972). This is a risk-adjusted performance measure representing the average

[8] The Sharpe ratio determines the excess returns that an investment can provide above the risk-free rate for each unit of volatility. Although this ratio is often used as a performance metric, it has several weaknesses; for example, it considers only risks and returns, and does not account for liabilities, differentiate between downside and upside possibilities, or perform well for non-normal distributions.

return on a portfolio or investment above or below that predicted by the capital asset pricing model, given the portfolio or investment beta and the average market return. These methodologies hold when the returns can be considered normally distributed. However, when the returns are asymmetric, these measures cease to capture the essential distributional features (see Pedersen et al., 2002). Thus, other approaches need to be used.

Since the classic Sharpe ratio cannot properly evaluate assets exhibiting non-normal distribution, several alternative performance measures have been suggested in the literature (some are still under debate). Thus, some ratios, such as the Omega ratio, Sortino ratio, Kappa ratio, upside potential ratio, Calmar ratio, Sterling ratio, Burke ratio, excess return on VaR, conditional Sharpe ratio, and modified Sharpe ratio (the one analyzed in this work) have been proposed to replace the Sharpe ratio and overcome some of its pitfalls. In a comprehensive review, (Eling & Schuhmacher, 2007) analyze and compare 13 different performance measures Pedersen & Rudholm-Alfvin (see also 2003, that can be used to select a risk-adjusted performance measure). However, all these metrics must be used with care since the evaluation of returns has been shown to be influenced by the choice of performance measure (Zakamouline, 2010). In this work, we consider the modified Sharpe ratio based on modified VaR, that is, the VaR computed using CF transformation.

The methods to compute VaR or determine the distribution quantiles have already been closely studied by academicians as well as practitioners since the introduction of VaR into the current banking system (For a comprehensive review of the methods, see Christoffersen, 2012; Jorion, 2007). A strand of the literature focuses on how to measure VaR. For example, Linsmeier & Pearson (2000), Duffie & Pan (1997), and Engle & Manganelli (1999) are general papers on measuring VaR. More specific studies have examined the primary methods, such as the Monte Carlo simulation by Pritsker (1997), Johnson transformations by Zangari (1996b), CF expansions by Zangari (1996a) and Fallon (1996), Solomon-Stephens approximation by Britten-Jones & Schaefer (1999), saddle-point approximations by Feuerverger & Wong (2000), and extreme value theory by Longin (2000). The theoretical properties of VaR evaluation have been reported by Artzner et al. (1999), Cvitanić & Karatzas (1999), and Wang (1999). Studies have also considered VaR optimization for portfolio or risk reduction. Gourieroux et al. (2000) is an appealing work on the sensitivity of VaR.

A small strand of the literature has sought to improve VaR estimation through the use of CF expansion. Pichler & Selitsch (1999) made a comparative study of the following five VaR methods in the context of portfolio and options: Johnson transformation, variance–covariance analysis, and the three CF expansions of second, fourth, and sixth orders. They concluded that the sixth order CF expansion is the best among the analyzed approaches. In this regard, we can also refer to the works of Mina & Ulmer (1999) and Feuerverger

& Wong (2000). Jaschke (2001) considered the properties of CF expansion and its underlying assumptions in the context of VaR, specifically focusing on non-monotonicity of the distribution function, where convergence cannot be guaranteed. [9] Jaschke discussed how the conditions for applying the CF approach make its use difficult in practice (we discuss the points in this paper). However, he demonstrated that when a dataset complies with the required conditions, the accuracy of CF expansion is generally more than sufficient to satisfy one's needs; moreover, it is faster than the other approaches. More recently, citepeling2007does analyzed and compared 13 different performance measures Pedersen & Rudholm-Alfvin (see also 2003, to select a risk-adjusted performance measure). However, all these metrics should be used with care because the evaluation of returns has been found to be influenced by the choice of performance measure (Zakamouline, 2010). Amédée-Manesme et al. (2015) used CF expansion and the so-called rearrangement procedure to calculate the direct real estate VaR. They calculated a rolling VaR over time for real estate returns and showed how CF expansion can be used to adequately account for the non-normality of returns in commercial real estate returns. Finally, Amédée-Manesme et al. (2019) proposed the use of response surface methodology to correct the CF parameters.

In this work, we consider the modified Sharpe ratio. This ratio is similar to the standard Sharpe ratio, is popular among practitioners, exhibits an explicit form, and is based on VaR–a metric proposed by regulators (Basel II and III, or Solvency II). It also relies on the first four moments (these are easy to understand and analyze), and is straightforward to compute and interpret.

3 The CF expansion

This study shows how to properly use the modified Sharpe ratio based on $mVaR$ risk measurement. The $mVaR$ measurement is based on CF expansion. However, CF expansion should be used with caution, especially with regard to two points: the validity domain of CF expansion must be carefully checked, and the quantile function (cumulative distribution function) could be non-monotone and must be rearranged.

The $mVaR$ calculation depends on the CF expansion. In short, CF expansion transforms naïve Gaussian quantiles by the skewness and kurtosis coefficients chosen to characterize the true distribution. This expansion is a simple polynomial function based on the corresponding unit normal quantile Taylor series (for more precision, see Stuart & Ord, 2009), where the coefficients of each resulting term are functions of the true distribution moments under consideration. CF expansion is useful because it allows for obtaining more accurate

[9] For a detailed discussion, see the chapter by Jaschke and Jiang in Härdle (2009).

results compared to those obtained using the central limit theorem (CLT) approximation. For a demonstration of the greater accuracy provided by CF expansion compared to CLT approximation, see, for example, Chernozhukov et al. (2010b).

For any $\alpha \in (0,1)$, the upper αth-quantile of F_n is defined as $q_n(\alpha) = inf\{x : F_n(x) \geq \alpha\}$, where F_n denotes the cumulative distribution function of $\xi_n = (\sqrt{n}/\sigma)(\bar{X} - \mu)$, and $\bar{X}$ is the sample mean of independent and identically distributed observations $X_1, \ldots, X_n$. If z_α denotes the upper αth-quantile of $N(0,1)$, the fourth-order CF expansion can be expressed as

$$q_n(\alpha) = z_\alpha + \frac{1}{6\sqrt{n}}(z_\alpha^2 - 1)S + \frac{1}{24n}(z_\alpha^3 - 3z_\alpha)(K-3) - \frac{1}{36n}(2z_\alpha^3 - 5z_\alpha)S^2 + o(n^{3/2}), \tag{4}$$

where S and K are the skewness and kurtosis of the observation X_i, respectively.

More simply, we denote the CF quantiles by $z_{CF,\alpha}$, to obtain the following expression for a normalized CF quantile at the probability level α:

$$z_{CF,\alpha} = z_\alpha + \frac{1}{6}(z_\alpha^2 - 1)S + \frac{1}{24}(z_\alpha^3 - 3z_\alpha)(K-3) - \frac{1}{36}(2z_\alpha^3 - 5z_\alpha)S^2, \forall \alpha \in (0,1), \tag{5}$$

where S and K denote the skewness and kurtosis coefficients of the true distribution, respectively. The corresponding modified CF quantile then becomes

$$q_{CF,\alpha} = \mu + z_{CF,\alpha}\sigma, \forall \alpha \in (0,1), \tag{6}$$

and the expression for VaR becomes

$$VaR_{CF,\alpha} = -q_{CF,\alpha} \forall \alpha \text{ such that } q_{CF,\alpha} < 0. \tag{7}$$

One can straightforwardly see that in the presence of an underlying Gaussian distribution ($S = 0$ and $K = 3$), equation (5), will reduce to the Gaussian quantile (and CF expansion will obviously be used when the distribution is normal).

CF expansion will thus approximate the quantile of a true distribution by using higher moments (skewness and kurtosis) of that distribution to adjust for the distribution's non-normality. Since the true distribution moments can be estimated in the standard manner by the sample skewness S and sample kurtosis K obtained from the data, these can be substituted into equation (5) to estimate the unknown quantiles (VaR) of the true distribution. CF expansion thus allows for considering higher-order distributional characteristics when performing quantile computation, so that the risky assets exhibiting non-normal distribution can be treated accurately. The CF approach thus offers

several advantages, such as its relative easy to implement and the skewness and kurtosis it allows for in VaR estimation.

As suggested, the CF approach leads to approximations closer to the true law compared to the traditional Gaussian approach, often to a dramatic degree. This is illustrated below in figure 1 for a chi-squared distribution with five degrees of freedom. Obviously, the CF methodology results cannot be better than the true distribution, but they do approach the true distribution far more closely than the Gaussian approximation. From the closeness of the CF approximations to the theoretical distribution, one can appreciate the power of this tool.

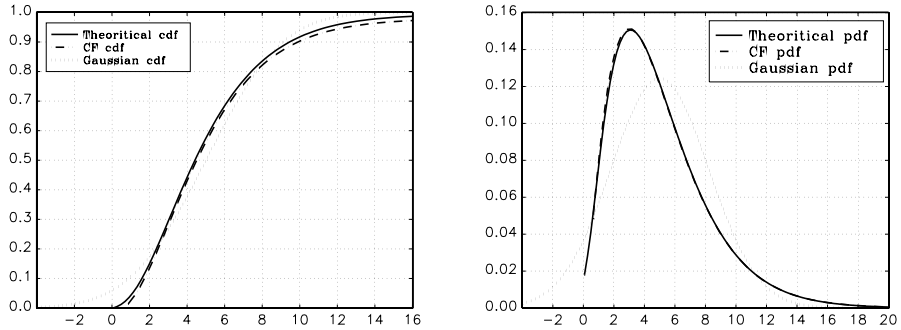

Fig. 1: CF approximation for a chi-squared distribution with five degrees of freedom $\mu = 5$; $\sigma^2 = 10$; $S = 1.27$; $K = 5.4$

Although CF expansion has proven to be a useful technique, the permitted values of the true distribution' moments have constraints by which the CF approximation itself leads to a true distribution. Relation (5) in general gives a non-monotonic character to z_{CF}; that is, the true distribution' ordering of the quantiles is not preserved, violating the basic condition that must be satisfied for the resulting CF approximation to be a proper cdf. Barton & Dennis (1952), Draper & Tierney (1973), and Maillard (2012), among others, study the CF expansion validity domain. Monotonicity requires a non-negative derivative of $z_{CF,\alpha}$ relative to z_α, which leads to the following constraint, implicitly defining the CF expansion validity (D) domain. :[10]

$$\frac{S^2}{9} - 4\left(\frac{K-3}{8} - \frac{S^2}{6}\right)\left(1 - \frac{K-3}{8} - \frac{5S^2}{36}\right) \leq 0. \tag{8}$$

[10] For example, inequality (8) implies a kurtosis coefficient higher than 3 (a positive excess of kurtosis), indicating a leptokurtic distribution. Thus, unadjusted CF expansion is not appropriate in the presence of thin tails.

Thus, if $(S, K) \in D$, the CF quantile function is monotonic, and otherwise, the CF method is inapplicable. Indeed, in this case, the quantile at a higher threshold can be smaller in absolute terms than that at a lower threshold ($\mid q_{\alpha_1} \mid < \mid q_{\alpha_2} \mid \forall \alpha_1 > \alpha_2$), which is obviously undesirable for any cumulative distribution function, and even less desirable when used for risk measurement. In practice, this constraint is rarely considered because S and K are generally considered small in finance. As stated above, the literature is generally silent on the CF definition domain.

For instance, consider a distribution with a skewness coefficient of 0.8 and kurtosis of 2. These parameters correspond to a thin-tailed and right-skewed distribution, respectively. Since these parameters do not belong to the D validity domain, the $z_{CF,\alpha}$ quantile function is not monotonic. However, by applying the rearrangement procedure to this quantile function, we obtain $\tilde{z}_{CF,\alpha}$, the corrected Gaussian quantile CF transformation. Focusing on the less than 25% probabilities, figure 3 shows the impact of this procedure. Note that it is not always possible to compute the non-rearranged CF probability density function because it could result in negative probabilities.

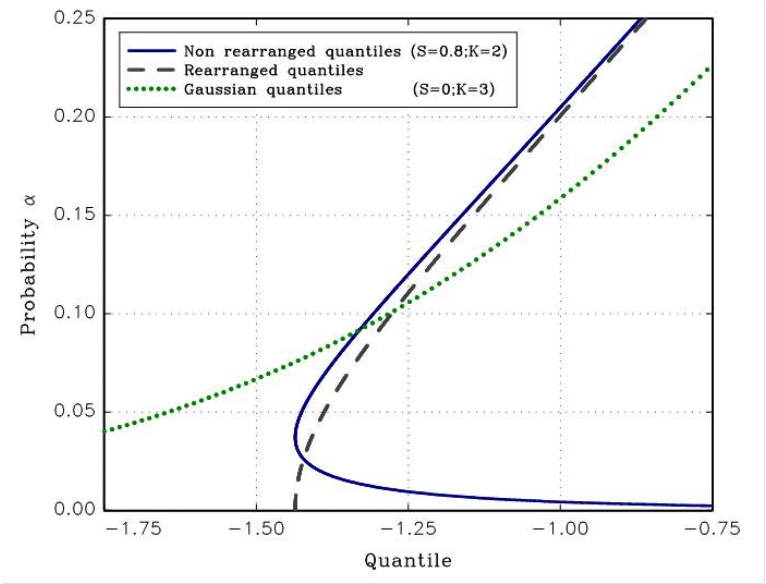

Fig. 2: The rearrangement procedure ($\alpha < 25\%$)

Furthermore, note that the discrepancy between two quantile functions, the rearranged and non-rearranged ones, is most noticeable for the smallest prob-

abilities, that is, the most important ones for VaR computation.[11]

To solve this issue, Chernozhukov et al. (2010a,b) proposed a procedure called increasing rearrangement to restore monotonicity. This procedure has been used in (Amédée-Manesme et al., 2015) to compute the VaR for real estate investment (whose returns exhibit abnormal behavior). Simply stated, rearrangement is a sorting operation: given the values of a dataset, the procedure simply sorts the values in an increasing order. Thus, the rearranged function is created (the procedure is broadly presented in Appendix B). As these authors demonstrate, the rearrangement procedure alters the non-monotone approximations such that they become monotonic. In our problem, this would correspond to the ascending sorting of quantile function $q_{CF,\alpha}$. Furthermore, and most importantly, as Chernozhukov et al. (2009) demonstrated, besides restoring monotonicity, rearrangement improves the estimation properties of the approximation. Indeed, Figure shows how the non-rearranged CF quantiles function leads to a rather poor estimation of the ideal quantiles. However, the rearranged quantiles are very close to the theoretical ones, clearly showing how the rearrangement can improve estimation quality. The argument is presented in Chernozhukov et al. (2010b). The improvement occurs because the rearrangement results in monotonicity, which necessarily brings the originally non-monotone approximations closer to the true monotone target function.[12] This improvement occurs because the rearrangement necessarily brings the non-monotone approximations closer to the true monotone target function.

This work uses the rearrangement procedure, which results in the proper use of CF expansion. We show how the proper use of CF expansion in the Sharpe ratio leads to different ranking in some cases.

4 Application

In the previous sections, we introduced the SR and mSR performance metrics. We showed the difficulties in proper computation of mSR by relying on $mVaR$, and introduced a solution—the so-called rearrangement. Our aim here is merely to apply our methodology to various distributions and illustrate how these metrics react. More precisely, we are interested in the parameters, and not in the distribution. Indeed, since SR and mSR are based on the first two and four moments of the distribution respectively, we will concentrate on that

[11] See the first figure in Chernozhukov et al. (2010b). Note that the non-rearranged quantile function might be even more severely non-monotonic (and therefore provide poorer distribution function approximations) than the one presented in figure 3. Note that $\tilde{z}_{CF,0.001} = -1.4$, whereas $z_{CF,0.001}$, which is equal to -0.3, is clearly biased.

[12] Note that the above example showed improved accuracy rearrangement that is guaranteed to restore monotonicity of the ideal distribution approximation and also improve the accuracy of that approximation compared to that achieved without rearrangement.

point.

In this section, we analyze the impact of various distributions. First, we simulate an almost normal distribution. The intuition is that moving away from this situation toward either skewness or kurtosis would significantly change the result. Second, we set the symmetry to 0 (the normal distribution value) and allow the kurtosis to move. Finally, we set the kurtosis to 3, and the simulated distribution will be asymmetric.

Our approach is particularly appropriate when the studied distribution is not normal. However, it remains relevant even in the presence of a normal distribution. Indeed, in this case, the skewness and kurtosis are set to 0 and 3, respectively, and equation 5 will add to the Gaussian quantile (z_α). Thus, determining the return distribution is not key to the process. One must be able to determine the first four moments of the dataset. Furthermore, the transformed $mVaR$ performance measures can contribute because all the distribution moments are taken into account and quantile estimation is more precise. More specifically, the further we move away from the normal case, the more important the improvement of our approach.

Our aim here is to compare the use of the following three performance measures: the return, Sharpe ratio (or more simply, volatility), and modified Sharpe ratio. More precisely, we focus on how the modified Sharpe ratio is computed. We therefore consider three variations in the modified Sharpe ratio—and thus five performance measures—that use a different method to compute $mVaR$: the modified Sharpe ratio computed under the normality assumption (hence, only with μ and σ), denoted as $NmSR$, the standard modified Sharpe ratio (with CF), denoted as mSR_{CF}, and the rearranged modified Sharpe ratio, ($mSR_{CF,\text{reag}}$).

The methodology consists of simulating 100 points of various distributions for 20 potential assets (which are to be compared) 1000 times. The simulated distribution replaces the empirically observed return series. This methodology is more convenient in our context than considering empirical data. Indeed, we can control the impact of the different parameters of interest, in our case, mainly S and K. We then compute the five performance metrics from the generated distributions and determine the corresponding ranks.

To capture the performance measures' degree of proximity, we use the rank correlation as an indicator of the proximity. All rank correlations can then be synthesized into a matrix (the rank correlation matrix). By construction, all the simulated distributions lead to one single rank correlation matrix, like a point estimation. In order to estimate all the correlation distributions, we have to replicate this random experiment several times. The average of these simulated matrices converges to the expected rank correlation matrix. Moreover, the replications allow us to estimate the cdf (or the pdf) of each matrix

element. Rank correlation can be used to measure the relationship between the different rankings of the same variable. Thus, the rank correlation coefficient measures the degree of similarity between the two rankings. In this study, we rely on Spearman's rank correlation coefficient. If two measures are conceptually equivalent, the theoretical rank correlation is equal to 1. This value becomes -1 in case of perfectly opposite ranks. A value of 0 corresponds to the situation where the two metrics are not linked at all.

Finally, in the next subsections, we compare the rankings obtained with the return, the SR, the mSR computed with the normal assumption ($NmSR$), and the mSR computed with the CF (mSR_{CF}) and finally corrected with the rearrangement procedure ($mSR_{CF,\text{reag}}$) for five different types of distributions: a nearly normal, a very leptokurtic (with skewness fixed to 0), a slightly asymmetric, and two highly asymmetric distributions.

Furthermore, note that the use of the Sharpe ratio and modified Sharpe ratio requires one to determine a risk-free rate (i) and a confidence level (or threshold) (ii).

i The choice of risk-free rate is an interesting issue for academicians and practitioners: the risk-free rate is often presumed to be given and/or easy to obtain. From a theoretical perspective, the risk-free rate is the rate of return of an investment with zero risk over a specified period of time. However, in reality, a risk-free rate does not exist, because all investments carry some amount of risk.

   In practice, both academics and practitioners use government security rates as the risk-free rates. In the context of this study, we consider the case of a foreign investor investing internationally and having the choice of a risk-free asset (national or local investors may face more limitations on their choice of risk-free asset). We choose the risk-free rate of 2% because an international investor can find assets with very limited risk for an annual return of about 2%.

ii Using the modified Sharpe ratio raises another issue, the choice of confidence level (the choice may appear a bit arbitrary). In particular, the choice of confidence level may have a strong impact on the determination of VaR, especially when the distribution exhibits high kurtosis. Risk models based on VaR presume the confidence level chosen. However, in practice, the justification of a particular confidence level is very difficult. Most textbooks illustrate VaR using a confidence level of 5% because it makes the comparison of two standard deviations easier. Several academic papers use a confidence level of 1% or 0.5%; these values are recommended respectively by the Basel and Solvency regulations. More recently, VaR is computed at increasingly lower levels. Indeed, with the development of more precise and improved databases, the extreme value theory, block maxima models,

and peaks over threshold models are increasingly used, and very low levels of VaR are becoming more common (see, for instance, Abad et al., 2014).

The choice of confidence level is not the primary topic of this paper, but we need to note that VaR computations under normal assumption give the same ranking irrespective of the threshold level. This will differ if the normal assumption is removed. This paper analyzes the sensitivity of the threshold on metrics (and therefore rankings).

4.1 Nearly Gaussian case

In this section, we simulate a normally (almost) distributed series (a purely normal distribution is not interesting in our case), with the distribution slightly deviating from the normal kurtosis and skewness set to 0. The variation in kurtosis will be automatically generated by sampling. With a sample size of 1000, we can ensure K values close to the Gaussian value.

Table 1 confirms the intuition. All the rankings, whatever be the metrics, are almost the same as for a correlation around 1. Since there is no rearrangement (or no need to rearrange), mSR_{CF} and $mSR_{CF,\text{reag}}$ are perfectly equal: the rankings are the same.

$\mathbf{R}$	SR	$NmSR$	$mSR_{\mathbf{CF}}$	$mSR_{\mathbf{CF,reag}}$
1.000	0.994	0.994	0.978	0.978
0.994	1.000	1.000	0.998	0.998
0.994	1.000	1.000	0.998	0.998
0.978	0.998	0.998	1.000	1.000
0.978	0.998	0.998	1.000	1.000

Note: The correlations are similar, as with other values of α.

Table 1: Rank Correlation Matrix — $\alpha = 0.001$, Gaussian Case

4.2 From $NmSR$ to mSR_{CF}: symmetric distribution with varying kurtosis

In order to simulate a perfectly symmetric distribution, we randomly generate a sample with length equal to one half of the sample size value, denoted hereafter as n, and the other half obtained being symmetric to 0. Thus, for an even sample with size n, we generate $n/2$ pseudo-random numbers $(x_1, \ldots, x_{n/2})$; the whole sample is by construction $(x_1, \ldots, x_{n/2}, -x_1, \ldots, -x_{n/2})$. This ensures a skewness equal to 0. Note that this result holds whatever be the distribution used for simulation. μ is set to 0.03. The standard deviation is set 0.2 (or 20% of volatility). The values of the simulated kurtosis can differ very

much, but can never be 3 (higher or smaller).

Several comments can be drawn from Tables 2 to 5. First, the correlation between $NmSR$ and mSR_{CF} are henceforth not the same. This result is as expected since the simulated distribution is not henceforth Gaussian. Second, this difference increases as the threshold decreases. Indeed, in this section, the distribution is symmetric ($S = 0$) and only the kurtosis varies. As the kurtosis mainly impacts the tails, the correlation is lower for a lower threshold. $\alpha = 0.05$ is not small enough to observe fatter tails. This latter comment show why the confidence level should be computed at a very low level, and not at 5%, as usually done in textbooks to compare the well-known normal distribution (Acerbi & Tasche, 2002, among others criticized the choice of 5%). Finally, these tables demonstrate how mSR_{CF} is more suitable than the standard Sharpe ratio and $NmSR$, whatever be the distribution. In fact, when the distribution can be considered as normally distributed, mSR_{CF} gives the same results as the other metrics and can better capture the difference when the distribution is not anymore normally distributed. Thus, this metrics could be a better choice.

R	SR	$NmSR$	$mSR_{\mathbf{CF}}$	$mSR_{\mathbf{CF,reag}}$
1.000	0.012	0.012	0.019	0.019
0.012	1.000	1.000	0.990	0.990
0.012	1.000	1.000	0.990	0.990
0.019	0.990	0.990	1.000	1.000
0.019	0.990	0.990	1.000	1.000

Table 2: Rank Correlation Matrix — $\alpha = 0.05$, symmetric distribution, varying K

R	SR	$NmSR$	$mSR_{\mathbf{CF}}$	$mSR_{\mathbf{CF,reag}}$
1.000	0.012	0.012	0.004	0.004
0.012	1.000	1.000	0.820	0.819
0.012	1.000	1.000	0.820	0.819
0.004	0.820	0.820	1.000	1.000
0.004	0.819	0.819	1.000	1.000

Table 3: Rank Correlation Matrix — $\alpha = 0.01$, symmetric distribution, varying K

From a computation of the correlation distribution in Figure 3, we find the correlation distributions not symmetric, but highly skewed to the left. This is more pronounced for a lower α; this is so because the difference between SR and mSR is more pronounced deeper in the tail (in this section, the distribution is symmetric, with only the kurtosis varying). Note that this effect tends to decrease when the number of assets increases (see Figure 9b in Appendix

R	SR	$NmSR$	$mSR_{\mathbf{CF}}$	$mSR_{\mathbf{CF,reag}}$
1.000	0.012	0.012	0.002	0.002
0.012	1.000	1.000	0.709	0.709
0.012	1.000	1.000	0.709	0.709
0.002	0.709	0.709	1.000	1.000
0.002	0.709	0.709	1.000	1.000

Table 4: Rank Correlation Matrix — $\alpha = 0.005$, symmetric distribution, varying K

R	SR	$NmSR$	$mSR_{\mathbf{CF}}$	$mSR_{\mathbf{CF,reag}}$
1.000	0.012	0.012	-0.003	-0.003
0.012	1.000	1.000	0.505	0.505
0.012	1.000	1.000	0.505	0.505
-0.003	0.505	0.505	1.000	1.000
-0.003	0.505	0.505	1.000	1.000

Table 5: Rank Correlation Matrix — $\alpha = 0.001$, symmetric distribution, varying K

C). However, the distribution of correlations tends to become less correlated when the number of assets is very low, as illustrated in Figure 9a in Appendix C, where the graphic is built with only five assets. Indeed, when few assets are considered, the chance of having a strong correlation is obviously lower than that when numerous assets are taken into account.

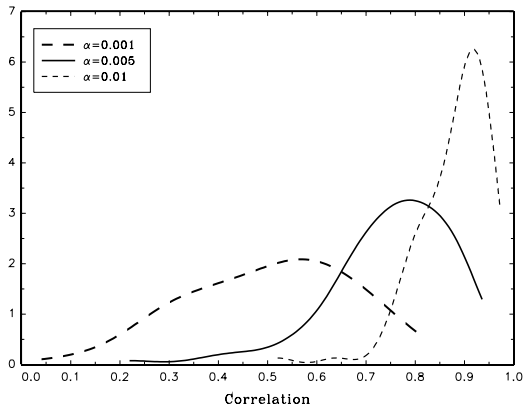

Fig. 3: Distributions of correlations between $NmSR$ and mSR_{CF}, $S = 0$.

4.3 From mSR_{CF} to $mSR_{CF,\text{reag}}$: asymmetric distribution with varying kurtosis

4.3.1 Example of non-monotone quantile and positive VaR

Note that we cannot compute mSR with too low thresholds for a comparison. Indeed, since CF expansion does not give the monotone function of a quantile, the quantile corresponding to a low threshold may be positive, but the corresponding VaR may not exist (VaR is a loss and cannot be positive). In these cases, rearrangement is the key. To illustrate this, we compute Figure 4 using a particular example, where the skewness and kurtosis are set to 1.8 and 3, respectively (the distribution is right-skewed). This graph shows how the quantile function resulting from a non-rearranged CF expansion can lead to a non-monotone function. This result is obviously misleading and false, since a lower threshold VaR may be higher than a higher threshold VaR. For instance, for $\alpha = 5\%$, we have $q_{0.05} = -1.645$. The CF quantiles are respectively $q^{CF}_{0.05} = -1.072$ and $q^{CF,reag}_{0.05} - 1.283$ after rearrangement.

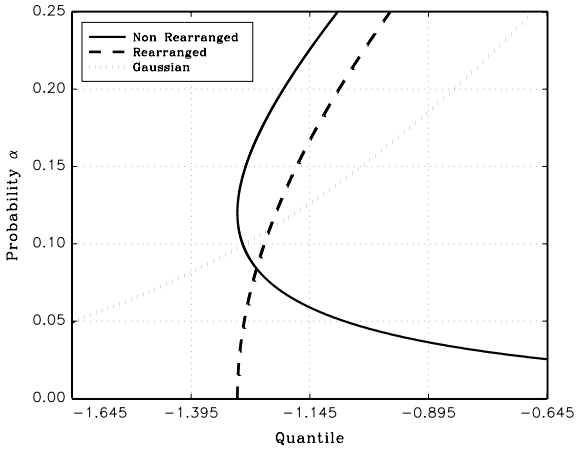

Fig. 4: Normal CF and rearranged CF quantiles functions with $S = 1.8$ and $K = 3$

In Figure 4, when the two curves cannot be differentiated, the rearrangement procedure would be useless because the ordering of quantiles is not affected by the CF procedure (in such cases, the ordering of quantiles is already increasing). However, by differentiating the two curves (particularly in the lower tails), we find the importance of rearranging (or sorting) the CF procedure results. In particular, it has considerable impacts on the rankings. This will be illustrated in the next section.

4.3.2 Asymmetric distribution, K=3

In this subsection, we simulate the skewness values uniformly in the range $[\,1.6; 2\,]$. The kurtosis is set to 3. Thus, the ranks will change when the difference between the two CF quantiles is sufficiently high. The values used for this example have no particular interest, except to illustrate the impact of the rearrangement. The estimated correlations displayed in Tables 6 and 7 are computed for $\alpha = 5\%$ and 4%, respectively. The rearrangement modifies the ranks compared to $NmSR$, but more importantly compared to mSR_{CF}. Note that the spread between the correlation ranks increases as the skewness values increase and decrease for smaller values. For instance, with $S \sim U[\,1.3; 2.3\,]$, the correlation drops to 0.327 (increasing the uniform range for skewness reduces the correlation). To conclude, note that with $S \sim U[\,0; 3\,]$, the correlation drops to 0.100.

For a α higher than 1, the correlation tends toward 1, whereas when α tends toward 0, the correlation does not exist anymore. Indeed, the VaR computed with CF is positive as the quantile is 0.935 instead of -1.539 after the rearrangement.

R	SR	$NmSR$	$mSR_{\mathbf{CF}}$	$mSR_{\mathbf{CF,reag}}$
1.000	0.004	0.004	0.004	0.006
0.004	1.000	1.000	0.982	0.673
0.004	1.000	1.000	0.982	0.673
0.004	0.982	0.982	1.000	0.680
0.006	0.673	0.673	0.680	1.000

Table 6: Rank Correlation Matrix — $\alpha = 0.05$, asymmetric distribution $(S \sim U[\,1.6; 2\,])$, $K = 3$

R	SR	$NmSR$	$mSR_{\mathbf{CF}}$	$mSR_{\mathbf{CF,reag}}$
1.000	0.004	0.004	0.005	0.010
0.004	1.000	1.000	0.997	0.611
0.004	1.000	1.000	0.997	0.611
0.005	0.997	0.997	1.000	0.609
0.010	0.611	0.611	0.609	1.000

Table 7: Rank Correlation Matrix — $\alpha = 0.04$, asymmetric distribution $(S \sim U[\,1.6; 2\,])$, $K = 3$

Finally, we simulated a negative skewness to illustrate how the rearrangement impacts the ranking of a negatively asymmetric distribution. To do so, the skewness values were simulated uniformly in the range $[\,-2; -1.6\,]$ with the kurtosis still set to 3. The results are presented in Table 8 for $\alpha = 0.3\%$. The

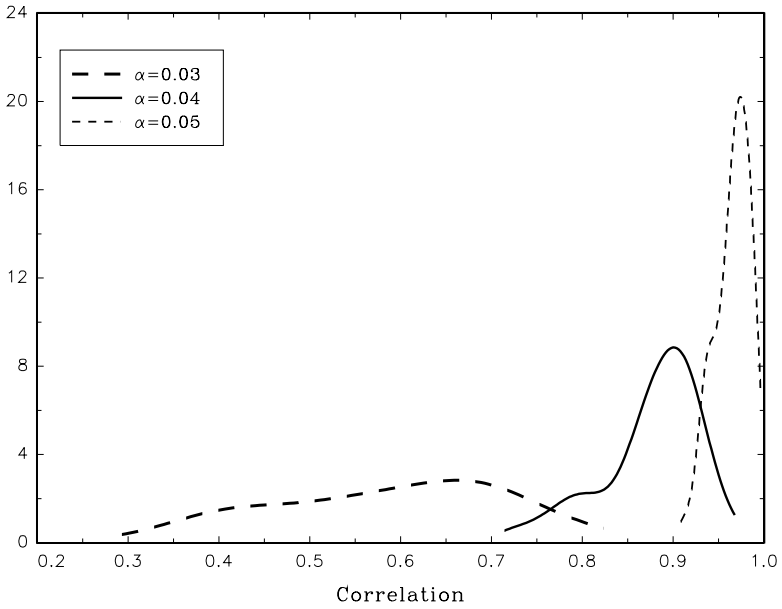

Fig. 5: Distributions of correlations between mSR_{CF} and $mSR_{\mathbf{CF},\text{reag}}$, asymmetric distribution

results are in line with the previous case and show the impact of the CF rearrangement procedure. The rank correlation is different between mSR_{CF} and $mSR_{CF,\text{reag}}$, but with lower difference. This is so because the rearrangement really begins from a much smaller threshold (α) than in the previous case. This is illustrated in Figure 6. Note that the non-rearranged CF quantile once again leads to a non-negative VaR. Therefore, the correlation between $NmSR$ and $mSR_{CF,\text{reag}}$ is higher than the correlation between $NmSR$ and mSR_{CF}, or mSR_{CF} and $mSR_{CF,\text{reag}}$. This comes from the signs of the quantiles, which are always the same for $NmSR$ and mSR_{CF} (given the thresholds computed here). All this shows the importance of the rearrangement procedure when needed.

$\mathbf{R}$	SR	$NmSR$	$mSR_{\mathbf{CF}}$	$mSR_{\mathbf{CF,reag}}$
1.000	-0.017	-0.017	-0.044	0.031
-0.017	1.000	1.000	0.390	0.339
-0.017	1.000	1.000	0.390	0.339
-0.044	0.390	0.390	1.000	-0.652
0.031	0.339	0.339	-0.652	1.000

Table 8: Rank Correlation Matrix — $\alpha = 0.003$, asymmetric distribution $(S \sim U[-2; -1.6])$, $K = 3$

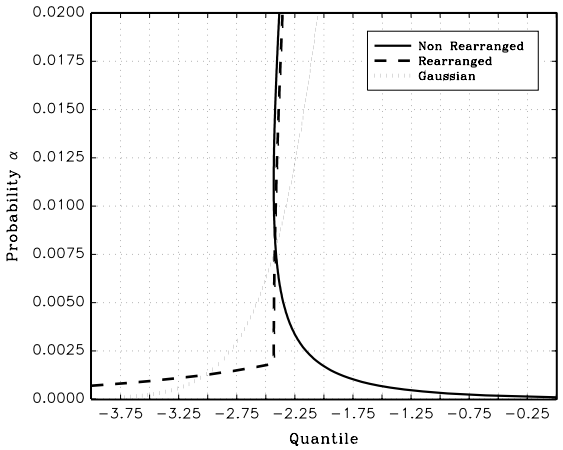

Fig. 6: Quantiles functions with $S = -1.8$ and $K = 3$

To conclude, the changes in rankings in most cases are relatively large, as shown by the correlation matrix. This underlines the robustness of our approach. All these observations call for some comments. First, the three mSR methods give somewhat different results when the distribution is not normal. This result contradicts the findings of Lee & Higgins (2009), who noted similar rankings in most instances for valuation-based property data in Australia using a similar approach (but without rearrangement and with a small deviation from the normal distribution). The difference in rankings is significant even for a distribution close to the normal. Second, the threshold level is important. Particularly for a low threshold, a mSR_{CF} without rearrangement should be banned from practice (unfortunately, its use is common) as it may lead to wrong and inaccurate results. Furthermore, the difference in correlation would increase as the threshold decreases. Finally, our proposed approach should be preferred, whatever be the distribution of returns, since it gives the same results as other approaches when the distribution is normal, but can also capture deviations from the normal.

5 Conclusion

The traditional Sharpe ratio approach presents some limitations, making its use tricky despite its popularity among practitioners. This ratio ignores the possible non-normality of returns and can cause investors to invest inappropriately in risky assets. The modified Sharpe ratio approach, however, can overcome these limitations. In particular, it relies on the modified VaR, a risk metric based on the entire distribution of the returns since its computation considers the third and fourth moments of distribution.

The modified Sharpe ratio is based on CF expansion, a useful technique that must be used with caution: the resulting distribution and quantile function approximations can be non-monotone and therefore give wrong results. This pitfall can be solved by using a rearrangement procedure introduced by (Chernozhukov et al., 2009, 2010b) that helps to properly compute the modified Sharpe ratio.

In this study, we showed how to properly use the modified Sharpe ratio (that is, by using the rearrangement procedure). We demonstrated and highlighted many results. First, we showed why our proposed approach should be preferred, whatever be the distribution of returns, as it gives the same results of the other standard approaches when the distribution is normal but can also capture any deviation from the normal. Second, the rearrangement procedure should be used particularly when the threshold is low (below 1%). Third, and finally, the rearranged modified Sharpe ratio becomes unavoidable as soon as the distribution deviates from the normal to correctly estimate the performance rankings.

The proposed methodology should be appealing to both practitioners and academics. Indeed, it is relatively easy to compute and facilitates the quantitative risk management of real estate transactions. The modified Sharpe ratio is a powerful tool to manage and to deal with portfolios exhibiting non-normal returns; it is also useful when the returns display normal patterns because the skewness and Kurtosis terms will cancel each other out. Our approach will therefore remain relevant even with a normal distribution. The ratio is therefore useful because many investments are volatile by nature and may not always be driven by a normal distribution. Finally, this work shows the way to many other applications and risk measurement, such as operational risk management or rare events.

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A Appendix: Skewness and kurtosis

Given the probability distribution $f(x)$ of the random variable X and a real-value added function $g(x)$, one defines the expectation $E[g(X)] = \int g(x)f(x)dx$, where the first moment is $\mu = E[X]$, and the higher central moments are defined as $\mu_n = E[(X - \mu)^n]$. The first task in almost all statistical analyses is to characterize the location and variability of a dataset. This is captured by the moments of orders one and two, usually called the mean μ and variance $\sigma^2 = \mu_2$, respectively. A further characterization of the data often includes the standardized moments of orders three and four, called skewness, $\gamma_1 = \mu_3/\sigma^3$, and kurtosis, $\beta_2 = \mu_4/\sigma^4$, respectively. These last two measures further describe the shape of a probability distribution. We briefly state the significance of these two last parameters below.

Skewness measures the symmetry or, more precisely, the lack of symmetry. A distribution or dataset is symmetric if it appears similar to the right and left of its center (the mean μ). The skewness of any symmetric distribution, such as a Gaussian distribution, is necessarily zero. Negative skewness coefficient values indicate skewed data to the left, whereas positive values indicate skewed data to the right. Skewness to the left means that the left tail of the distribution is long relative to the right tail.

Kurtosis indicates whether the data are peaked or flat relative to a normal distribution. That is, datasets with high kurtosis tend to have a distinct peak near the mean, then decline rather rapidly, and still show heavy tails. Datasets with low kurtosis tend to have a flat top near the mean, rather than a sharp peak. The kurtosis formula measures the degree of this peakedness; for instance, the kurtosis of a Gaussian distribution turns out to be 3.

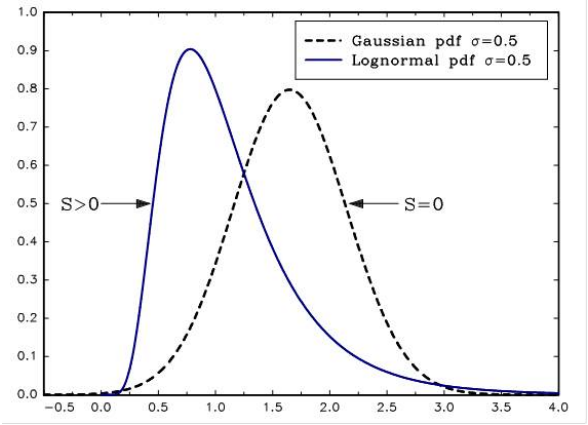


Fig. 7: Right-skewed distribution ($S = 1.75$)

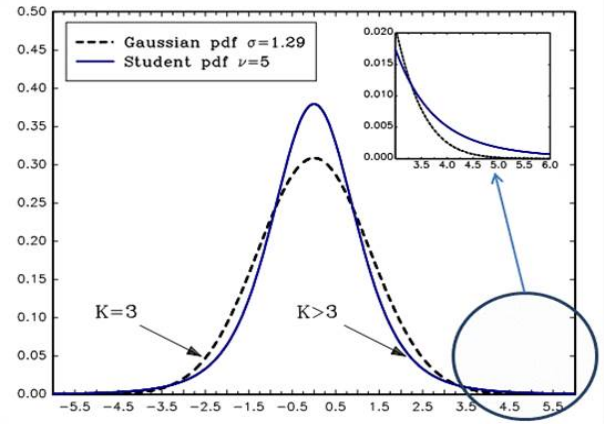


Fig. 8: Fat-tailed distribution ($K = 9$)

B Appendix: The Rearrangement Procedure

This paper applies a procedure called rearrangement, or, more precisely, increasing rearrangement. We use this procedure to restore the CF expansion monotonicity. The procedure is briefly described as follows.[13].

Rearrangements can be conveniently considered to be a sorting operation: given the values of a dataset, we simply sort the values in an increasing order. What is created is the rearranged function.

Following Chernozhukov et al. (2010a,b), we define the procedure more precisely as follows.

> "Let χ be a compact interval. Without loss of generality, we can conveniently take this interval to be $X = [0, 1]$. Let $f(x)$ be a measurable function mapping χ to K, a bounded subset of $\mathbb{R}$. Also, let $F_f(x) = \int_\chi 1\{f(u) \leq y\}\, du$ denote the distribution of $f(x)$ when X follows the uniform distribution on $[0, 1]$, and
>
> $$f^*(x) = Q_f(x) = inf\left\{y \in \mathbb{R} : F_f(y) \geq x\right\}$$
>
> be the quantile function of $F_f(y)$. Thus,
>
> $$f^*(x) = inf\left\{y \in \mathbb{R} : \left[\int_\chi 1\{f(u) \leq y\}\, du\right] \geq x\right\}.$$
>
> This function, f^*, is called the increasing rearrangement of the function f."

In our approach, this allows for respecting one of the basic conditions of the probability distribution function, monotonicity. Thus, the VaR becomes inversely proportional to the threshold, and, as expected, one has $VaR_{0.5\%} \geq VaR_{5\%}$.

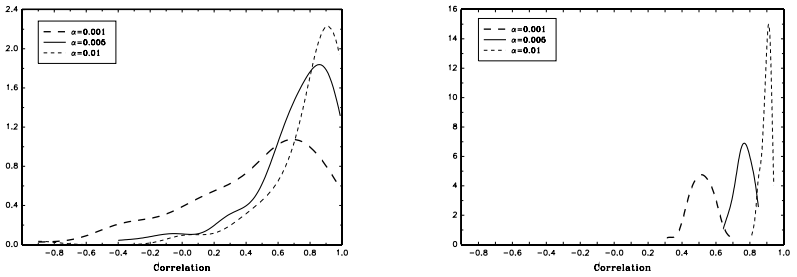
The rearrangement procedure also has a practical implication, demonstrated by Chernozhukov et al. (2010b): the resulting rearranged estimate has a smaller estimation error

[13] In mathematics, the notion of rearrangement derives from the notion of permutation, and is reported in the work of Bóna (2004). (Lorentz, 1953) can also be consulted

(in the Lebesgue norm) than does the original estimate whenever the latter is not monotone. This property is independent of the sample size and how the original approximation is obtained. Thus, the benefits of using a rearrangement procedure in our paper are both to obtain the estimates of the distribution satisfying the logically necessary monotonicity restriction, and also to obtain better approximation properties.

C Appendix: Impact of number of assets

The number of assets simulated may also have a strong impact on the relevance of the results. To illustrate the effect of number of assets, we compute Figure 9. Intuitively, the higher the number of assets, the lower is the dispersion of distribution. The results are thus more concentrated.



(a) 5 assets **(b)** 100 assets

Fig. 9: Correlation distributions according to the number of assets